Reconstruction of spatial resolution of multilayer position-sensitive detectors

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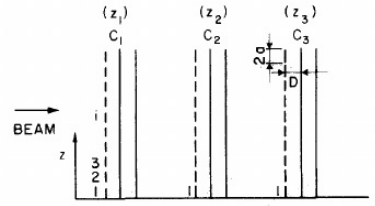
Content

Introduction

- Consider a series of detectors measuring the coordinates of a track.
- Measurement of only one coordinate is considered.
- Tests with one track in a time, assume detectors are 100%-efficient, no noise $\Rightarrow$ one hit per plane, no need of combinatorial track recognition, reconstruction etc.
- Needed resolutions of detectors with the least possible number of assumptions. No assumption of layers identity. Why unwanted?
  - Obvious differences in design, unintended (unknown) microscopic differences, different electronic interference in external and internal layers.
  - Detector studies.
  - Optimization of detector performance.
- If individual resolutions cannot be found, how about their average (squared average)?
- No reference detectors. True track coordinates and inclination angles unknown.
- The measurements of each track in different detectors (in "layers", in "detecting layers") can be fitted by a track model and residuals can be obtained.
- Tracks are straight here.
- Distributions of residuals are functions of distributions of errors.
  Variances of residuals are functions of variances of errors.
  How to obtain the latter from the former?
- If there are very many layers, the uncertainties of tracks may be negligible.
  Then one can equate the layer resolution to residual
  (most likely to the "exclusive" one).
  Here the case of a few layers is considered.

Introduction, what is known

- The assumption of identical layer resolution, only the proportionality coefficient is needed:
  G. Charpak, et al., NIM 167(1979)455,
  F. Piuz, et al., NIM 196(1982)451,
  A. Korytov, et al, NIM A 338(1994)375,
  and many others.



$R_z = \frac{z_1+z_3}{2} - z_2 + \text{misaligment} \Rightarrow$

"fwhm" of residuals is $2.36\sigma\sqrt{3/2}$. (F. Piuz et al, 1982)
Without a proof: the error propagation rules $\Rightarrow$
$V(z_i) \neq 0$, $\text{corr}(z_i, z_j) \neq 0$.

- The method of geometric mean: R.K. Carnegie et al, NIM A 538(2005)372, quotes:
  (Also D.C. Arogancia et al, NIM A 602(2009)403)
  Weights are inverse layer variances,
  which are unknown.

Let us assume a track consisting of N measurements with known values y_j, $1 \leqslant j \leqslant N$. The corresponding measured values x_j are distributed around the expected mean $\langle x_j \rangle = a + by_j$ with the standard deviations s_j, where a and b are the track parameters. To determine the resolution of one measurement i it is convenient to choose the coordinate system so that $y_i = 0$. In this case, the residual is given by $\delta_i = a - x_i$, where a can be determined from a least square fit to the track by either including (a^{in}) or excluding (a_i^{ex}) the measurement i. The residual δ_i will be distributed with a standard deviation σ_{δ_i} which is related to s_i, but depends on the coordinates (x_j, y_j) and weights $w_j = 1/s_j^2$ of all measurements.

Expanding, rearranging and collecting terms yields:

$$\sigma^2_{\delta_i^{\text{in}}} = s_i^2 \frac{D_i^{\text{ex}}}{D^{\text{in}}} \qquad \text{(A.7)}$$

and similarly

$$\sigma^2_{\delta_i^{\text{ex}}} = s_i^2 \frac{D^{\text{in}}}{D_i^{\text{ex}}}.$$

The quantities D^{in} and D_i^{ex} are fixed for a given layout and can be calculated to correct the resolution measured, however it is simpler to combine the last two expressions:

$$s_i^2 = \sigma_{\delta_i^{\text{in}}} \sigma_{\delta_i^{\text{ex}}}. \qquad \text{(A.8)}$$

- Actually, useful only as an approximation for very many layers (weights are unimportant) or for equal layer resolutions (weights are canceled).
  MC tests in Ref. [T. Alexopoulos et al., JINST, 9 (2014) P01003]:
  "The geometric mean method produces accurate results when the test and reference detectors have the same characteristics."

Introduction, what is known (2)

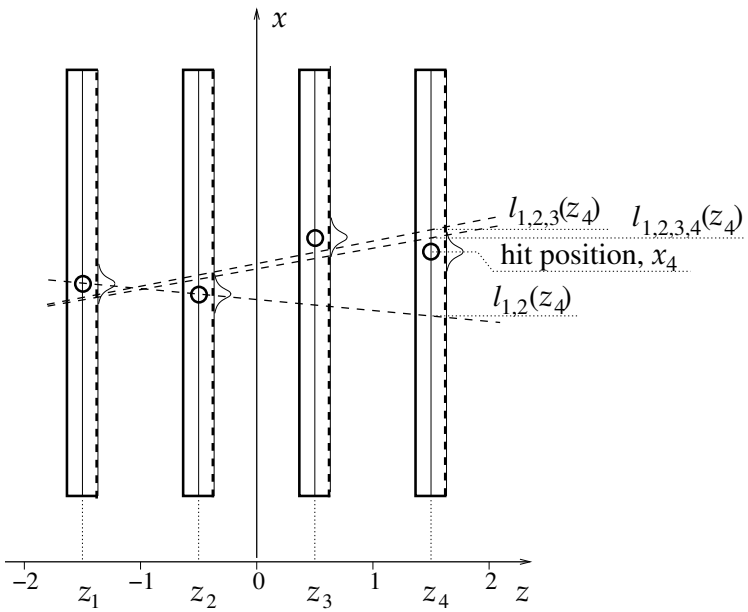
- The use of the correction factors "calculated from diagonal elements of "Hat" matrix" [I. A. Golutvin et al., Physics of Part. Nucl. Let., 7(2010) 355].

> Coming to the layer spatial resolution one has to multiply the residuals for each layer by the correction factors calculated from the diagonal elements of the "Hat matrix" [7]. For ME1/1 CSC layers the correction factors are 1.45 for layers 1 and 6, 1.19 for layers 2 and 5,

  The "Hat" matrix is from [A.C. Rencher et al., Linear Models in Statistics, 2008, page 228].
  In Rencher notation: $\mathbf{y} = \mathbf{X}\boldsymbol{\beta} + \boldsymbol{\epsilon}$, $\boldsymbol{\epsilon} = \sigma^2\mathbf{I}$, $\hat{\boldsymbol{\epsilon}} = \boldsymbol{y} - \boldsymbol{X}\hat{\boldsymbol{\beta}}$, $\hat{\boldsymbol{\beta}} = (\mathbf{X'X})^{-1}\mathbf{X'y}$,
  $\hat{\mathbf{y}} = \mathbf{X}\hat{\boldsymbol{\beta}} = \mathbf{X}(\mathbf{X'X})^{-1}\mathbf{X'} = \mathbf{Hy}$, $\hat{\boldsymbol{\epsilon}} = (\mathbf{I} - \mathbf{H})\boldsymbol{y} = (\mathbf{I} - \mathbf{H})\epsilon$.
  The resolutions are assumed identical (although can be calculated separately).
- Claim of reconstructing 4 resolutions:
  J. Bortfeldt et al., IEEE Trans. Nucl. Sci. 59 (2012) 1252.
  J. Bortfeldt, Springer Theses, 2015;
  Not reproduced.
- Obtaining resolution from sum of residuals (with assumption of identical layers):
  Unbiased estimator $\Rightarrow$ can be averaged by many tracks.
  $\sigma^2 = \frac{Q^2_{\min}}{N-r}$ [F. James, Statistical Methods..., 2006, Section 8.4.1, page 185],
  [A.C. Rencher et al., Linear Models in Statistics, 2008, page 131] ,
  [Kendall et al. Section 19.9].

Main notations

Example of 4-layer detector
(like Cathode Strip Chamber)



N is the number of layers (in the plot $N = 4$).
z_i are positions of detecting layers.
x_i, $i = 1, 2, 3, 4$ are measured coordinates of hits.
$x_{t,i}$ are true positions of hits.
ϵ_i are errors of measurements: $x_i = x_{t,i} + \epsilon_i$.
$l_{1,2\ldots}(z)$ is the position of the straight line fitted by layers 1, 2... at z.
r_i are residuals $x_i - l_{1,2\ldots}(z_i)$.
$E(\xi)$ is the expectation of any value ξ (which can be x_i, ϵ_i, etc.).
$\sigma(\xi)$ is the standard deviation of any value ξ.
$V(\xi) = \sigma(\xi)^2$ is the variance of any value ξ.
$\mathrm{cov}(\xi_1, \xi_2) = E[(\xi_1 - E(\xi_1))(\xi_2 - E(\xi_2))] =$
$= \mathrm{corr}(\xi_1, \xi_2)\sigma(\xi_1)\sigma(\xi_2)$ is the covariance and the correlation.
$\boldsymbol{\xi}$, $E(\boldsymbol{\xi})$, $\sigma(\boldsymbol{\xi})$, $V(\boldsymbol{\xi})$ are vectors with N components.

No systematic shifts and no electric cross talks between layers $=>$

- the expectation $E(\epsilon_i) = 0$,
- the correlations $\mathrm{corr}(\epsilon_i, \epsilon_j) = 0$,

But $x_{t,i}$ are correlated! $=>$ x_i are correlated as well!

Error propagation rules (informal notation):

- $E\left(\sum_i^N a_i\xi_i\right) = \sum_i^N a_i E(\xi_i)$,
- $V\left(\sum_i^N a_i\xi_i\right) = \sum_i^N a_i^2 V(\xi_i) + 2\sum_i^N \sum_{j=i+1}^N a_i a_j \ \mathrm{cov}(\xi_i, \xi_j)$,
- $\mathrm{cov}\left(\sum_i^N a_i\xi_i, \sum_j^N b_j\xi_j\right) = \sum_i^N a_i \sum_j^N b_j \ \mathrm{cov}(\xi_i, \xi_j)$.

The straight line fit

The straight line: $l(\boldsymbol{x}, z_i) = b(\boldsymbol{x}) + a(\boldsymbol{x})z$.
Find the parameters by minimization of $M = \sum_{i=1}^{N} w_i \big(x_i - l(\boldsymbol{x}, z_i)\big)^2$, w_i is the weight.
The optimal line: $\hat{l}(\boldsymbol{x}, z_i) = \hat{b}(\boldsymbol{x}) + \hat{a}(\boldsymbol{x})z$.
Denote $s = \sum_{i=1}^{N} w_i$, $\overline{x} = \sum_{i=1}^{N} w_i x_i / s$, $\overline{z} = \sum_{i=1}^{N} w_i z_i / s$, $\overline{xz} = \sum_{i=1}^{N} w_i x_i z_i / s$, $\overline{z^2} = \sum_{i=1}^{N} w_i z_i^2 / s$, $D(z) = \overline{z^2} - \overline{z}^2$.
Minimization of M results in

$$\hat{a}(\boldsymbol{x}) = \frac{\overline{xz} - \overline{x}\ \overline{z}}{D(z)}, \qquad \hat{b}(\boldsymbol{x}) = \overline{x} - a\overline{z} = \frac{\overline{x}\,\overline{z^2} - \overline{xz}\ \overline{z}}{D(z)}\ .$$

These formulas can be found in many sources, for example,
[V. K. Grishin et al., Math. Threatment and Interp. of Phys. Exper., 1988]

This is the special case of linear method of least squares or linear regression:
$x(z) = \sum_{j=1}^{N} a_j f_j(z)$, or $\boldsymbol{x} = F\boldsymbol{a}$, where:

$$F = \begin{pmatrix} f_1(z_1) & f_2(z_1) & \dots & f_k(z_1) \\ f_1(z_2) & f_2(z_2) & \dots & f_k(z_2) \\ \dots & \dots & \dots & \dots \\ f_1(z_N) & f_2(z_N) & \dots & f_k(z_N) \end{pmatrix}, \quad W = \begin{pmatrix} w_1 & 0 & \dots & 0 \\ 0 & w_2 & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & w_N \end{pmatrix},$$

$$\hat{\boldsymbol{a}} = (F^{\mathsf{T}} W F)^{-1} F^{\mathsf{T}} W \boldsymbol{x}.$$

In out case $f_1(z) = 1$, $f_2(z) = z$.

Residuals

Distribution of residuals for each layer i: $r_i = x_i - \hat{l}(\boldsymbol{x}, z_i)$.

Zero expectation, because otherwise better $\hat{b}(\boldsymbol{x})$ exists.

Its variance $V(r_i) = V(x_i - \hat{l}(\boldsymbol{x}, z_i))$ depends on known $\boldsymbol{w}$ and unknown $\boldsymbol{\epsilon}$ and unknown $\boldsymbol{x}_\mathrm{t}$. Application of the error propagation rules is useless.

r_i is a linear combination of components of the vector $\boldsymbol{x}$:

$r_i(\boldsymbol{x}) = x_i - \hat{l}(\boldsymbol{x}, z_i) = \sum_{j=1}^{N} u_{ij} x_j$, where u_{ij} are constants that depend on $\boldsymbol{w}$ and $\boldsymbol{z}$, but do not depend on $\boldsymbol{x}$.

The residual calculated with true $\boldsymbol{x}_\mathrm{t}$ should be zero. Therefore, by construction:

$$r_i(\boldsymbol{x}_\mathrm{t}) = x_{\mathrm{t},i} - l(\boldsymbol{x}_\mathrm{t}, z_i) = \sum_{j=1}^{N} u_{ij} x_{\mathrm{t},j} = 0 \ .$$

Substituting $x_i = x_{\mathrm{t},i} + \epsilon_i$ into $r_i(\boldsymbol{x})$ we obtain

$$r_i(\boldsymbol{x}) = \sum_{j=1}^{N} u_{ij}(x_{\mathrm{t},j} + \epsilon_j) = \sum_{j=1}^{N} u_{ij} x_{\mathrm{t},j} + \sum_{j=1}^{N} u_{ij}\epsilon_j = \sum_{j=1}^{N} u_{ij}\epsilon_j = \epsilon_i - l(\boldsymbol{\epsilon}, z_i).$$

Cf. quotation from Rencher, slide 5 (there it is without the weight matrix).

Conclusion: in any residual as a function of $\boldsymbol{x}$, we can substitute $\boldsymbol{x}$ by $\boldsymbol{\epsilon}$.

After that we can apply error propagation rules, taking into account $\mathrm{cov}(\epsilon_i, \epsilon_j) = 0$:

$$V(r_i) = \sum_{j=1}^{N} u_{ij}^2 V(\epsilon_j) = \sum_{j=1}^{N} h_{ij} V(\epsilon_j), \quad \mathrm{cov}(r_i, r_j) = \sum_{k=1}^{N} u_{ik} u_{jk} V(\epsilon_k)$$

We can draw different lines with different $w_i \neq 0$ for the same layer, but this does not seem to give any advantage. The trivial choice $w_i = 0 \vee w_i = 1$ is enough. Covariances do not give additional information. The matrix H consists of elements h_{ij}:

$$V(\boldsymbol{r}) = HV(\boldsymbol{\epsilon}) \quad \Rightarrow \quad V(\boldsymbol{\epsilon}) = H^{-1} V(\boldsymbol{r}) \ .$$

Corrections for multiple scattering and for longitudinal fluctuations of ionization

Suppose that

a) The multiple scattering occurs only in the planes of layers;

b) There is no lateral shift of the trajectory;

c) The standard deviation of the scattering angle ϕ_i, denoted by $\sigma(\phi)$, in any single layer i is so low that $\tan\sigma(\phi_i) \approx \sigma(\phi_i)$;

d) The inclination of tracks with respect to the detector plane is so low that $\sigma(\phi_i)$ does not significantly depend on it.

The addition caused by scattering in the layer $k = j + 1$ for residual in the layer i is $\sum\limits_{m \geq k} \sigma^2(\phi_m)(z_i - z_m)^2$. If $k = j + 2$ and if the scattering occurs also in the intermediate layer $j + 1$, the addition is $\sigma^2(\phi_{j+1})(z_i - z_{j+1}) + \sum\limits_{m \geq k} \sigma^2(\phi_m)(z_i - z_m)^2$.

Generalization for continuous multiple scattering and for inclined trajectories is possible.

In order to obtain clean $\sum_{j=1}^{N} u_{ij}^2 V(\epsilon_j)$, these additions should be subtracted from experimentally found $V(r_i)$.

Similar addition for longitudinal fluctuations of ionization can be $\sum_{j=1}^{N} u_{ij}^2 \frac{1}{N_{\mathrm{tracks}}} \sum\limits_{\mathrm{all\ tracks}} (\sigma_{\mathrm{lf},j} \tan(\theta_{\mathrm{track}}))^2$, where $\sigma_{\mathrm{lf},i}$ is the standard deviation for layer i per unity length of projection of the segment of track inside the sensitive volume to x-axis.

One can take into account many other effects in similar way.

Shapes of residual distributions

Any linear combination of Normally distributed random numbers is also a Normal distribution with the variance which is also a linear combination of variances with squared coefficients.

Assume that each ϵ_i is distributed by a Gaussian $G(\epsilon_i, 0, \sigma_i)) = \frac{1}{\sqrt{2\pi}\sigma} \exp\left(-\frac{1}{2}\frac{\epsilon_i^2}{\sigma_i^2} \right)$,

Then r_i is distributed according to $G(r_i, 0, \sum_{j=1}^{N} u_{ij}^2 \sigma_i^2)$.
Let ϵ_i be distributed by a sum of Gaussians $\sum_k \rho_{ik} G(\epsilon_i, 0, \sigma_{ik})$,
where the sum of weights $\sum_k \rho_{ik} = 1$. The variance of resulting r_i will be a sum of all possible combinations of variances of N Gaussians:

$$V(r_i) = \sum_{k_1} \rho_{1,k_1} \sum_{k_2} \rho_{2,k_2} \cdots \sum_{k_N} \rho_{N,k_N} \sum_{j=1}^{N} u_{ij}^2 \sigma_{jk_j}^2 \ .$$

Geometric means, inclusive and exclusive residuals

Consider residual for layer i obtained with the line fitted
by all N layers (including i, "inclusive") and
by $N-1$ layers excluding layer i ("exclusive").
Weights w_i are identical for inclusive and exclusive cases (except for layer i; they can differ for different layers).
Following [R.K. Carnegie, et al, NIM A 538(2005)372] choose z such that $z_i = 0$.
Denote the non-normalized weighted average of any value v_i by two bars: $\overline{\overline{v}} = \sum_i w_i v_i$.

Therefore, the "ordinary" average $\overline{v} = \overline{\overline{v}}/s$. Also let $\sum_{k\neq i} w_k = s_i^{(\mathrm{ex})}$. Then:

$$\left(s_i^{(\mathrm{ex})}\right)^2 D_i^{(\mathrm{ex})}(z) = s_i^{(\mathrm{ex})}\overline{\overline{z^2}} - \overline{\overline{z}}^2, \quad s^2 D(z) = s\overline{\overline{z^2}} - \overline{\overline{z}}^2.$$

$$s^2 D(z) = \left(s_i^{(\mathrm{ex})}\right)^2 D_i^{(\mathrm{ex})}(z) + w_i \overline{\overline{z}}^2 \ .$$

$$r_i(\boldsymbol{x}) = x_i - b_i = x_i - \frac{\overline{x}\ \overline{z^2} - \overline{xz}\ \overline{z}}{D(z)} = \frac{x_i s^2 D(z) - \overline{\overline{x}}\ \overline{\overline{z^2}} + \overline{\overline{xz}}\ \overline{\overline{z}}}{s^2 D(z)}$$

$$= \frac{x_i \left(s_i^{(\mathrm{ex})}\right)^2 D^{(\mathrm{ex})}(z) - \sum_{j\neq i} w_j x_j \left(\overline{\overline{z^2}} - z_j \overline{\overline{z}}\right)}{s^2 D(z)} = \frac{x_i \left(s_i^{(\mathrm{ex})}\right)^2 D^{(\mathrm{ex})}(z) - \sum_{j\neq i} w_j C_j x_j}{s^2 D(z)},$$

$$C_j = \overline{\overline{z^2}} - z_j \overline{\overline{z}}\ , \qquad r_i^{(\mathrm{ex})}(\boldsymbol{x}) = \frac{x_i \left(s^{(\mathrm{ex})}\right)^2 D^{(\mathrm{ex})}(z) - \sum_{j\neq i} w_j C_j x_j}{(s^{(\mathrm{ex})})^2 D^{(\mathrm{ex})}(z)}$$

Numerators are identical!

Geometric mean with optimal weights

Denote the geometric mean $V_i^{(\mathrm{gm})}(r) = \sqrt{V(r)V(r_i)^{(\mathrm{ex})}}$.

$$V_i^{(\mathrm{gm})}(r) = \frac{\left(s_i^{(\mathrm{ex})}\right)^4 \left(D_i^{(\mathrm{ex})}(z)\right)^2 V(\epsilon_i) + \sum\limits_{j\neq i} w_j^2 C_j^2 V(\epsilon_j)}{s^2 D(z)(s_i^{(\mathrm{ex})})^2 D_i^{(\mathrm{ex})}(z)} \ .$$

The structure is the same as for the ordinary variances of residuals: the linear combination of resolution variances $\Rightarrow$ can be handled similarly.
According to [R.K. Carnegie, et al, NIM A 538(2005)372] and [D.C. Arogancia, et al, NIM A 602(2009)403] the weights w_i should be equal to inverse variance $w_i = 1/V(\epsilon_i)$ (optimal for the least squares method). Then,

$$V_i^{(\mathrm{gm})}(r) = \frac{\left(s_i^{(ex)}\overline{\overline{z^2}} - \bar{\bar{z}}^2\right)^2 V(\epsilon_i) + \sum\limits_{j\neq i} w_j C_j^2 w_i V(\epsilon_i)}{(s_i\overline{\overline{z^2}} - \bar{\bar{z}}^2)(s_i^{(ex)}\overline{\overline{z^2}} - \bar{\bar{z}}^2)}$$

$$= \frac{\left(s(s-w_i)\overline{\overline{z^2}}^2 - (2s-w_i)\bar{\bar{z}}^2\overline{\overline{z^2}} + \bar{\bar{z}}^4\right) V(\epsilon_i)}{s(s-w_i)\overline{\overline{z^2}}^2 - (2s-w_i)\bar{\bar{z}}^2\overline{\overline{z^2}} + \bar{\bar{z}}^4} = V(\epsilon_i).$$

This result is beautiful, but useless, because in order to obtain residuals with weights $w_i = 1/V(\epsilon_i)$, we have already to know these very resolutions $\sqrt{V(\epsilon_i)}$, which we want to obtain. Numerical tests show that an iteration procedure with remaking the residuals with previously obtained weights is not useful too.

Geometric mean with unity weights

Another interesting special case is $w_i = 1$, for which

$$V_i^{(\mathrm{gm})}(r) = \frac{\left((N-1)\overline{\overline{z^2}} - \overline{\overline{z}}^2\right)^2 V(\epsilon_i) + \sum\limits_{j\neq i}\left(\overline{\overline{z^2}} - z_j\overline{z}\right)^2 V(\epsilon_j)}{\left(N\overline{\overline{z^2}} - \overline{\overline{z}}^2\right)\left((N-1)\overline{\overline{z^2}} - \overline{\overline{z}}^2\right)} \ .$$

Let us assume that we have a telescope of detectors with the same resolution $V(\epsilon_j) = B$ and we want to study the tested detector i having possibly different resolution. Then, equation simplifies to

$$V_i^{(\mathrm{gm})}(r) = \frac{\left((N-1)\overline{\overline{z^2}} - \overline{\overline{z}}^2\right) V(\epsilon_i) + \overline{\overline{z^2}}B}{\left(N\overline{\overline{z^2}} - \overline{\overline{z}}^2\right)} \ .$$

This allows us to determine $V(\epsilon_i)$ provided that B is known or if B is a function of $V(\epsilon_i)$. If $B = V(\epsilon_i)$

$$V_i^{(\mathrm{gm})}(r) = V(\epsilon_i) \ ,$$

The Monte-Carlo simulations in Ref. [T. Alexopoulos et al., JINST, 9 (2014) P01003] seem to confirm that this formula is correct for equal resolutions and not accurate for non-equal resolutions, but unfortunately this work like many others do not specify which weights were used for track fitting. So we can only assume that the weights were unity by default. My simulations with unity weights bring about to the same conclusion. Refs. [Carnegie,Arogancia] also do not comment on what weights should be used to calculate the residuals in practical applications of the equation $V_i^{(\mathrm{gm})}(r) = V(\epsilon_i)$. Note that the same results can be easily obtained from any single residual:

$$V(\epsilon_i) = \frac{V(r_i)}{\sum_{j=1}^{N} u_{ij}^2}.$$

Reconstruction of resolution, three detecting layers

For 3 detecting layers all equations are linearly dependent. There are no combination of equations for residuals, correlations and geometric means with arbitrary weights and arbitrary z-positions, which allows one to obtain anything else.

For example, 3 layers, unity weights and $z_1 = -1$, $z_2 = 0$, $z_3 = 1$.
Obviously, all lines are linearly dependent.

$$H = \begin{pmatrix} \frac{1}{36} & \frac{1}{9} & \frac{1}{36} \\ \frac{1}{9} & \frac{4}{9} & \frac{1}{9} \\ \frac{1}{36} & \frac{1}{9} & \frac{1}{36} \end{pmatrix}$$

Layer permutations:

The order of flayers: $\begin{pmatrix} \text{II} & \text{I} & \text{III} \\ \text{I} & \text{II} & \text{III} \\ \text{I} & \text{III} & \text{II} \end{pmatrix}$ $H = \begin{pmatrix} \frac{1}{9} & \frac{1}{36} & \frac{1}{36} \\ \frac{1}{9} & \frac{4}{9} & \frac{1}{9} \\ \frac{1}{36} & \frac{1}{36} & \frac{1}{9} \end{pmatrix}$ $H^{-1} = \begin{pmatrix} 10 & -\frac{1}{2} & -2 \\ -2 & \frac{5}{2} & -2 \\ -2 & -\frac{1}{2} & 10 \end{pmatrix}$

The idea to check layer permutations was proposed by N. V. Gruzinsky.

All resolutions can be obtained, if permutations or moving is possible.
It is enough to move just one layer to 3 different positions (in total).

Reconstruction of resolution, four detecting layers

For 4 detecting layers all equations are linearly dependent.
This was obtained by Reduce computer algebra system.
Layer permutations or moving allow the user to obtain separate resolutions, as in the case of 3 layers.
But less detained information can be obtained without permutations.
Consider 4 layers with unity gaps, 3-layer straight lines, fitted with unity weights, and residuals by the fourth layers ("exclusive" residuals).

$$H = \begin{pmatrix} 1 & \frac{16}{9} & \frac{1}{9} & \frac{4}{9} \\ \frac{16}{49} & 1 & \frac{4}{49} & \frac{1}{49} \\ \frac{1}{49} & \frac{4}{49} & 1 & \frac{16}{49} \\ \frac{4}{9} & \frac{1}{9} & \frac{16}{9} & 1 \end{pmatrix}$$

This matrix is singular.
Subtract the second line of H from the other lines $i = 1, 3, 4$ with factors $h_{i,2}/h_{2,2}$, subtract the third line of H from the other lines $i = 1, 2, 4$ with factors $h_{i,3}/h_{3,3}$, subtract the first line of H from the other lines $i = 2, 3, 4$ with factors $h_{i,1}/h_{1,1}$.
In addition, subtract the second line of H with factor $h_{3,4}/h_{2,4}$ from the third line.
The same operations with $V(\boldsymbol{r})$ are assumed. The resulting H-matrix is

$$\begin{pmatrix} \frac{200}{477} & 0 & 0 & \frac{200}{477} \\ 0 & 1 & 0 & \frac{-1}{3} \\ 0 & \frac{2385}{2401} & \frac{2385}{2401} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Reconstruction of resolution, four detecting layers (2)

Equating the last element of modified similarly $V(\boldsymbol{r})$ to zero one can obtain the dependence of $V(r_4)$ on the other three residuals:

$$V(r_4) = V(r_1) - \frac{49}{27}(V(r_2) - V(r_3)).$$

The identical $h_{1,1} = h_{1,4}$ and $h_{2,2} = h_{2,3}$ means that it is possible to obtain the average squared resolutions:

$$\frac{V(\epsilon_1) + V(\epsilon_4)}{2} =$$

$$= \frac{1431V(r_1) - 2548V(r_2) + 49V(r_3)}{1200} = \frac{477(V(r_1) + V(r_4)) - 833(V(r_2) + V(r_3))}{800},$$

$$\frac{V(\epsilon_2) + V(\epsilon_3)}{2} =$$

$$= \frac{-459V(r_1) + 1372V(r_2) + 539V(r_3)}{1200} = \frac{-153(V(r_1) + V(r_4)) + 637(V(r_2) + V(r_3))}{800},$$

$$\frac{1}{4}\sum_{i=1}^{4} V(\epsilon_i) = \frac{81(V(r_1) + V(r_4)) - 49(V(r_2) + V(r_3))}{400}.$$

The last result can be obtained also by finding appropriate values of parameters α_i in the linear combination

$$\sum_{i=1}^{4} \alpha_i V(r_i) = \boldsymbol{\alpha}^{\mathsf{T}} H V(\boldsymbol{\epsilon}) = \frac{1}{4}\sum_{i=1}^{4} V(\epsilon_i), \quad H^{\mathsf{T}}\boldsymbol{\alpha} = \frac{1}{4}1 .$$

with taking into account symmetry $\alpha_1 = \alpha_4$ and $\alpha_2 = \alpha_3$.

Reconstruction of resolution, four detecting layers (3)

If to add the first line of the final presented H-matrix with a factor (1/6)/(200/477) to the second line and to assume that $V(\epsilon_1) \approx V(\epsilon_4)$,
it is possible to reconstruct the resolution of the second and the third layer separately.
If $V(\epsilon_1) \neq V(\epsilon_4)$, the result is not exact, but is still reasonable.

The "magic" ratio $H_{24}/H_{22} = -1/3$ appeared in H-matrix after described operations was obtained by many the other methods of line drawing for unity weights and equal spacing, including even the use of geometrical means instead of residuals for constructing H-matrix.

If $z_2 - z_1 = z_4 - z_3$ (configuration is symmetric), one can obtain similar results (with different numerical parameters).

If $z_2 - z_1 \neq z_4 - z_3$, exact averages cannot be calculated.
If the difference is not large, approximate averages can be obtained.
One of the methods is by fitting vector $\boldsymbol{\alpha}$ from the previous slide. If can be done algebraically or numerically. Details are in future paper.

Reconstruction of resolution, five detecting layers

- For any 4 detector layers we can write 3 independent equations.
- Consider two combinations of 4 layers taken from 5 layers, such that the second combination includes the layer which was not included in the first one.
- For the second combinations of layers write equations that include the replaced layer.
- Then any equation of the second combination is not linearly dependent on equations of the first combination because of the appearance of extra layer.
- Therefore we can obtain 5 and even more independent equations.

For example, two-layer straight lines provide 195 non-singular systems of equations. Obviously, there are much more independent equations for 6- and more-layer detectors. It is reasonable to use simple "exclusive" residuals with number of equations being equal to the number of layers.
The example of inverse matrix for equally spaced five layers:

$$H^{-1} = \begin{pmatrix} 1.746 & -1.749 & -0.187 & 0.701 & -0.653 \\ -0.571 & 1.578 & -0.022 & -0.260 & 0.229 \\ -0.046 & -0.017 & 1.026 & -0.017 & -0.047 \\ 0.228 & -0.260 & -0.022 & 1.578 & -0.571 \\ -0.654 & 0.701 & -0.188 & -1.748 & 1.746 \end{pmatrix}.$$

Conclusion

1. A method for reconstructing individual resolutions of detecting layers in multi-layer detectors is developed.
2. The individual layer resolutions can be obtained for 3- and 4-layer detectors only if there is a possibility to move the layers.
3. If the layers cannot be moved, but the 4-layer detector is symmetric, the following values can be obtained:
   a) The average squared resolution of four layers;
   b) The average squared resolutions of layers 1 and 4, as well as 2 and 3;
   c) If the resolutions of layers 1 and 4 are assumed to be equal, the individual resolutions of all layers (assuming the first and the fourth identical) can be obtained.
4. If the symmetry is slightly violated in the 4-layer detector, approximate estimates of all values mentioned in the previous item can be obtained.
5. The individual resolutions can be obtained for 5- and more-layer detectors.
6. All these results can be calculated by either residuals or geometrical means of inclusive and exclusive residuals. The geometric means of residuals, as well as correlations of residuals, do not produce any additional information. The geometric means obtained with unity weights are equal to the layer resolution only if the detector layers have the same resolution.
7. If some layer resolution can be obtained algebraically, then the shape of the corresponding error distribution can be obtained by fitting.

Thank you for your attention!